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An outcome space approach for generalized convex multiplicative programs

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Abstract This paper addresses the problem of minimizing an arbitrary finite sum of products of two convex functions over a convex set. Nonconvex problems in this form constitute a class of generalized convex multiplicative problems. Convex analysis results allow to reformulate the problem as an indefinite quadratic problem with infinitely many linear constraints. Special properties of the quadratic problem combined with an adequate outer approximation procedure for handling its semi-infinite constrained set enable an efficient constraint enumeration global optimization algorithm for generalized convex multiplicative programs. Computational experiences illustrate the proposed approach.

Keywords Global optimization · Multiplicative programming · Convex analysis · Indefinite quadratic programming · Numerical methods

1 Introduction

This paper is concerned with the problem of minimizing an arbitrary finite sum of products of two convex functions over a convex set. Nonconvex problems in this form constitute a class of generalized convex multiplicative problems. In convex multiplicative programming one is interested in obtaining globally optimal solutions for the nonconvex problem of minimizing an arbitrary finite product of convex functions over a convex set [14]. A traditional technique in multiplicative programming is to project the multiplicative problem in the *outcome space*, that is, in the real space where the vector of convex functions that constitute the multiplicative objective has its image.

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A number of different approaches for solving generalized multiplicative problems in the outcome space have been proposed. An outcome space approach for minimizing sums or products of ratios of linear functions is presented in [6] and [7]. The outcome space is defined by a mapping that associates to each original ratio a new variable. Upper and lower bounds on the optimal solution of the nonconvex problem are obtained by solving a sequence of linear programming subproblems.

Outer approximation is also employed in [16], which extends the multiplicative programming approach introduced in [18] to the minimization of an arbitrary finite sum of products of two convex functions. By transforming the original objective the authors obtain and solve an equivalent concave minimization problem by a cutting plane algorithm. The parametric transformation proposed in [18] is extended in [11] to the minimization of the sum of a convex function and an arbitrary finite product of convex functions. This particular generalized multiplicative problem is rewritten as a quasiconcave minimization problem and solved by a conical branch-and-bound algorithm.

Some generalized convex multiplicative formulations are closely related to generalized fractional programming [27]. Generalized linear multiplicative and fractional programming problems are tackled in [13] using a precursor of the parametric transformation introduced in [18]. A parametric simplex algorithm for generalized linear fractional programming is proposed in [15]. A specialization for generalized linear fractional problems of the algorithm derived in [16] is carried out in [17].

Branch-and-bound techniques are also traditional in the field of generalized multiplicative programming. Affine and generalized affine multiplicative problems are treated in [26] by using a combination of a lower bounding procedure proposed by the authors in [25] and a new branching scheme. Branch-and-bound is used in [2] for globally minimizing a sum of ratios of nonlinear functions in its equivalent outcome space formulation. A simplicial branch-and-bound algorithm for minimizing the sum of ratios of linear functions is presented in [3]. A rectangular branch-and-bound algorithm for the global maximization of generalized concave multiplicative functions has been recently proposed in [4].

An extension of the outcome space convex multiplicative programming approach proposed in [21] is presented in this paper. As in [21] convex analysis results are used for outer approximating generalized convex multiplicative programs. However, the proposed extension introduces a remarkable difficulty. Whereas in the purely multiplicative case the coordination problem relies on solving a linearly constrained quasiconcave global minimization problem by vertex enumeration, in the generalized case the coordination demands the global solution of a linearly constrained indefinite quadratic problem. Despite its apparent difficulty, it is shown that characteristics such as small number of variables (when the problem is represented in the outcome space), small number of linear constraints (as the result of an effective outer approximation scheme), and a special property of the quadratic form, render the global solution of the coordination problem efficient by constraint enumeration. An additional geometric property and bounds on the ϵ-optimal solutions obtained by the proposed global optimization algorithm are also introduced.

The paper is organized as follows. In Sect. 2 the formulation and relevance of the intended class of generalized convex multiplicative problems are briefly discussed. In Sect. 3 the problem is reformulated in the outcome space as an indefinite quadratic problem with infinitely many linear constraints. Some aspects of the outer approximation scheme for solving the problem in the outcome space are detailed in Sect. 4 Computational experiences are developed and analysed in Sect. 5. Conclusions are presented in Sect. 6.

Notation. The set of all n-dimensional real vectors is represented as $\mathbb{R}^n$. The sets of all nonnegative and positive real vectors are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. Inequalities are meant to be componentwise: given $x, y \in \mathbb{R}^n$, then $x \geq y$ ($x - y \in \mathbb{R}^n_+$) implies $x_i \geq y_i,\ i = 1, 2, \ldots, n$. Accordingly, $x > y$ ($x - y \in \mathbb{R}^n_{++}$) implies $x_i > y_i,\ i = 1, 2, \ldots, n$. The standard inner product and the Euclidean norm in $\mathbb{R}^n$ are denoted as $\langle x, y\rangle$ and $\|x\|$, respectively. The subset of boundary points of $\Omega \subset \mathbb{R}^n$ is denoted as $\partial\Omega$. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined on Ω, then $f(\Omega) := \{f(x) \ :\ x \in \Omega\}$. The symbol $:=$ means *equal by definition*.

2 Problem statement and overview

Consider the generalized convex multiplicative problem

$$(P_{\mathrm{M}}) \left| \begin{array}{l} \text{minimize } v(x) = f_1(x) + \sum_{i=1}^{r} f_{2i}(x) f_{2i+1}(x) \\ \text{subject to} \quad g_j(x) \leq 0,\ j = 1, 2, \ldots, p, \end{array} \right.$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ ($i = 1, 2, \ldots, 2r + 1$) and $g_j : \mathbb{R}^n \to \mathbb{R}$ ($j = 1, 2, \ldots, p$) are convex functions. As usual it is assumed that

$$\Omega := \{x \in \mathbb{R}^n \ :\ g_j(x) \leq 0,\ j = 1, 2, \ldots, p\} \tag{1}$$

is a nonempty compact (convex) subset of $\mathbb{R}^n$ and that each f_i is positive over Ω. Formulation (P_{M}) is representative of important mathematical programming problems. Given a quadratic objective function $v(x) = (1/2)\langle x, Qx\rangle + \langle c, x\rangle$, where $c \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ has rank equal to $r \leq n$, there exist linearly independent sets of n-dimensional vectors, $\{c^1, c^2, \ldots, c^r\}$ and $\{d^1, d^2, \ldots, d^r\}$, such that

$$v(x) = \langle c, x\rangle + \sum_{i=1}^{r} \langle c^i, x\rangle\langle d^i, x\rangle.$$

See [28] for a comprehensive discussion about decomposition of quadratic and bilinear forms that rely on the formulation (P_{M}).

The problem of minimizing a sum of ratios is another important application of generalized multiplicative programming. Suppose that $f_1, f_2, \ldots, f_r$ are convex and $h_1, h_2, \ldots, h_r$ are concave positive functions over $\Omega \subset \mathbb{R}^n$. Then each $1/h_i(x)$ is convex and positive over Ω and the fractional problem

$$(P_{\mathrm{R}}) \left| \begin{array}{l} \text{minimize } v(x) = \sum_{i=1}^{r} f_i(x)/h_i(x) \\ \text{subject to} \quad x \in \Omega \end{array} \right.$$

reduces to (P_{M}). Problem (P_{R}) includes the case where one or more ratios are not proper (that is, $h_i(x) = 1$ for some i), which describes the situation where the objective is to minimize a sum of absolute and relative terms.

Examples of algorithms that address problem (P_{R}) are found in [13] (for the case of linear ratios) and [2] (for the case of nonlinear ratios). Applications of fractional programming which rely on problems of the form (P_{R}) are surveyed in [27]. In particular, if each ratio f_i/h_i is a risk/profit measure, then by solving (P_{R}) one seeks a compromise solution for a (possibly weigthed) sum of risk-profit ratios. The nonlinear programming approach proposed in [23] for the synthesis of heat exchanger networks also relies on problems of the form (P_{R}), with each f_i/h_i being described by a linear fractional function and Ω by linear constraints.

A rectangular branch-and-bound algorithm for the problem of maximizing a generalized concave multiplicative function in the form considered in (P_{M}) has been recently proposed in [4]. It is worth observing that the problem considered in [4] is not equivalent to (P_{M}).

The objective function in (P_{M}) can be written as the composition $v(x) = u(f(x))$, where $u : \mathbb{R}^m \to \mathbb{R}$, $m = 2r + 1$, defined by

$$u(y) := y_1 + \sum_{i=1}^{r} y_{2i}\, y_{2i+1}, \tag{2}$$

is viewed as a particular aggregating function for the problem of minimizing the vector-valued objective $f := (f_1, f_2, \ldots, f_m)$ over Ω [29]. The *outcome space* associated with problem (P_{M}) is the projection of Ω onto $\mathbb{R}^m$ under f:

$$\mathcal{Y} := f(\Omega). \tag{3}$$

A solution $x^\star \in \Omega$ is an *efficient solution* of the multiplicative (multiobjective) problem (P_{M}) if there exists no other $x \in \Omega$ such that $f(x) \leq f(x^\star)$ and $f(x) \neq f(x^\star)$. The set of all efficient solutions of (P_{M}) is denoted as $\mathrm{effi}(\Omega)$. The positiveness of f over Ω implies that u is strictly increasing on $\mathcal{Y}$ and, in consequence, that any optimal solution of (P_{M}) is efficient.

It is known from the multiobjective programming literature [29] that if $x \in \Omega$ is an efficient solution of (P_{M}) then there exists $w \in \mathbb{R}^m_+$ such that x is also an optimal solution of the convex programming problem

$$(P_W) \left| \begin{array}{l} \text{minimize } \langle w, f(x) \rangle \\ \text{subject to } \quad x \in \Omega. \end{array} \right.$$

Conversely, let $x(w)$ denote any optimal solution of (P_W), given $w \in \mathbb{R}^m_+$. Then $x(w)$ is efficient if $w \in \mathbb{R}^m_{++}$. By defining

$$\mathcal{W} := \left\{ w \in \mathbb{R}^m_+ \;:\; \sum_{i=1}^{m} w_i = 1 \right\}, \tag{4}$$

the efficient set $\mathrm{effi}(\Omega)$ can be generated by solving (P_W) over $\mathcal{W}$. The characterization of optimal solutions of (P_{M}) as efficient solutions of its multiobjective programming counterpart has been explored in the convex multiplicative programming literature [1,12,21].

3 Outcome space formulation of (P_{M})

The *outcome space* formulation of (P_{M}) is

$$(P_{\mathcal{Y}}) \left| \begin{array}{l} \text{minimize } u(y) = y_1 + \displaystyle\sum_{i=1}^{r} y_{2i}\, y_{2i+1} \\ \text{subject to } \quad y \in \mathcal{Y}, \end{array} \right.$$

where $\mathcal{Y}$ is defined by (3).

The continuity of f and the compactness of Ω imply the compactness of $\mathcal{Y}$. Although the convexity of f and Ω do not entail the convexity of $\mathcal{Y}$, they do entail the connectedness of

$\mathcal{Y}$ by arcs: if $y^1, y^2 \in \mathcal{Y}$, then there exist $x^1, x^2 \in \Omega$ such that $y^1 = f(x^1)$, $y^2 = f(x^2)$ and the continuous mapping f satisfies $f(\alpha x^1 + (1-\alpha)x^2) \in \mathcal{Y}$ for all $\alpha \in [0, 1]$ [24].

The set of all efficient solutions in the outcome space is $\text{effi}(\mathcal{Y}) = f(\text{effi}(\Omega))$. It is readily seen that if $y \in \text{effi}(\mathcal{Y})$ then $y \in \partial\mathcal{Y}$. A practical representation of problem $(P_{\mathcal{Y}})$ is introduced in Theorem 1 on the basis of the following Lemma.

Lemma 1 *Given $y \in \mathbb{R}^m$, the system of inequalities $f(x) \leq y$ has a solution $x \in \Omega$ if and only if y satisfies*

$$\langle w, y\rangle \geq \min_{x\in\Omega} \langle w, f(x)\rangle \quad \textit{for all} \quad w \in \mathcal{W}.$$

Proof See, for example, [19]. □

Theorem 1 *Let $y^\star$ be an optimal solution of the problem*

$$(P_{\mathcal{F}}) \left| \begin{array}{l} \text{minimize } u(y) = y_1 + \displaystyle\sum_{i=1}^{r} y_{2i}y_{2i+1} \\ \text{subject to } y \in \mathcal{F}, \end{array} \right.$$

where

$$\mathcal{F} := \left\{ \underline{y} \leq y \leq \overline{y} \;:\; \langle w, y\rangle \geq \min_{x\in\Omega} \langle w, f(x)\rangle \text{ for all } w \in \mathcal{W} \right\} \tag{5}$$

and

$$\underline{y}_i := \min_{x\in\Omega} f_i(x) > 0 \quad \textit{and} \quad \overline{y}_i := \max_{x\in\Omega} f_i(x), \quad i = 1, 2, \ldots, m. \tag{6}$$

Then $y^\star$ is also an optimal solution of $(P_{\mathcal{Y}})$. In addition, $y^\star \in$ effi$(\mathcal{Y})$.

Proof As any $y = f(x),\ x \in \Omega$, is feasible for $(P_{\mathcal{F}})$, the feasible set of $(P_{\mathcal{F}})$ contains the feasible set of $(P_{\mathcal{Y}})$ and the optimal value of $(P_{\mathcal{F}})$ is a lower bound for the optimal value of $(P_{\mathcal{Y}})$. If $y^\star$ solves $(P_{\mathcal{F}})$, then by Lemma 1 there exists $x^\star \in \Omega$ such that $f(x^\star) \leq y^\star$ and, actually, $f(x^\star) = y^\star$. Otherwise ($f(x^\star) \leq y^\star$ and $f(x^\star) \neq y^\star$), the feasibility of $f(x^\star)$ for $(P_{\mathcal{F}})$ and the positivity of u on $\mathcal{F}$ would contradict the optimality of $y^\star$. Since $f(x^\star)$ is feasible for $(P_{\mathcal{Y}})$, one concludes that $y^\star$ also solves $(P_{\mathcal{Y}})$. The existence of another optimal solution y^0 such that $y^0 \leq y^\star$ and $y^0 \neq y^\star$ would contradict again the optimality of $y^\star$. Consequently, $y^\star \in \text{effi}(\mathcal{Y})$. □

Differently from $(P_{\mathcal{Y}})$, problem $(P_{\mathcal{F}})$ exhibits a convex feasible set. On the other hand, $(P_{\mathcal{F}})$ falls in the category of semi-infinite programming problems, as its feasible set includes a semi-infinite linear inequality system. Relaxation is a possible solution strategy in such context.

Generally speaking, relaxation consists in temporarily dropping all but a few constraints and solve a relaxed problem. If the relaxed problem is infeasible, so is the original. If an optimal solution of the relaxed problem satisfies all ignored constraints, the solution is also optimal for the original problem. If not, one or more violated constraints are incorporated into the previous constrained set and the procedure is repeated.

According to the relaxation procedure proposed in this paper only the most violated constraint of the semi-infinite linear inequality system in $\mathcal{F}$ is incorporated into the previous constrained set. The most violated constraint is determined by computing

$$\theta(y) := \max_{w\in\mathcal{W}} \phi_y(w), \tag{7}$$

where

$$\phi_y(w) := \min_{x\in\Omega} \langle w, f(x) - y\rangle. \tag{8}$$

Then y satisfies the semi-infinite inequality system if and only if $\theta(y) \leq 0$. Otherwise the maximizer on the right-hand side of (7) characterizes the most violated constraint.

The functions θ and ϕ_y exhibit a number of useful properties [21]. The function ϕ_y is concave over $\mathcal{W}$ and therefore $\theta(y)$ in (7) [respectively, $\phi_y(w)$ in (8)] is computed by solving a concave (respectively, convex) programming problem. The function θ is convex (and hence, continuous) over $\mathbb{R}^m$.

Concave programming problems of the form (7) and (8) can be efficiently tackled by the tangential approximation (linear programming) method discussed in [19] or by more sophisticated implementations of dual methods surveyed in [5]. The computational results reported in Sect. 5 were obtained by using the tangential approximation method.

An additional geometric property of θ is derived below.

Theorem 2 *$\theta(y)$, $y \in \mathbb{R}^m$, is the optimal value of the convex programming problem*

$$\left|\begin{array}{l} \text{minimize } \sigma \\ \text{subject to } f(x) \leq \sigma e + y, \\ \qquad\qquad\quad x \in \Omega, \end{array}\right. \tag{9}$$

where $\sigma \in \mathbb{R}$ and $e \in \mathbb{R}^m$ is the vector of ones.

Proof The dual problem of (9) is

$$\underset{w\in\mathbb{R}^m_+}{\text{maximize}} \quad \min_{x\in\Omega,\sigma\in\mathbb{R}} \{\sigma + \langle w, f(x) - \sigma e - y\rangle\}, \tag{10}$$

where $w \in \mathbb{R}^m_+$ is the vector of dual variables attached to the inequality constraints. The existence of the minimum imposes that $\sum_{i=1}^m w_i = 1$, and (10) reduces to

$$\underset{w\in\mathcal{W}}{\text{maximize}} \ \min_{x\in\Omega} \langle w, f(x) - y\rangle, \tag{11}$$

whose optimal value is $\theta(y)$. Since the primal problem (9) is convex, under mild constraint qualification assumptions [5] there is no duality gap and problems (9) and (11) have the same optimal value, $\theta(y)$. $\square$

Theorem 2 enables the following geometric interpretation of $\theta(y)$ for some $y \in \mathbb{R}^m$. Let $(x^\star, w^\star)$ be primal and dual optimal solutions of problem (9). By Theorem 2, $f(x^\star) \leq \theta(y)e + y$, and if $w^\star \in \mathbb{R}^m_{++}$ then the inequality becomes an equality. The case $\theta(y) > 0$ is more relevant for the analysis because the relaxation algorithm proposed in Sect. 4 generates a sequence of infeasible points ($y \notin \mathcal{F}$) converging to an optimal solution of $(P_\mathcal{F})$. In this case $\theta(y)$ is numerically equal to the infinity norm between y and $\mathcal{F}$.

4 Solving $(P_\mathcal{F})$ by relaxation

Consider the initial polytope $\mathcal{F}^0 := \{y \in \mathbb{R}^m \ : \ \underline{y} \leq y \leq \overline{y}\}$, where $\underline{y}$ and $\overline{y}$ are defined by (6). While obtaining $\underline{y}$ involves solving m convex programs, obtaining $\overline{y}$ requires m convex maximizations. Although $\overline{y}$ could be determined by effective global optimization methods [10], the simpler strategy of making $\overline{y}$ large enough in order that $\mathcal{F}^0$ contains an optimal solution of (P_M) has been adopted.

The minimum of u over $\mathcal{F}^0$ is attained at $y^0 = \underline{y}$ and, by using (7), one generally concludes that $\theta(y^0) > 0$, that is, y^0 is not feasible for $(P_{\mathcal{F}})$. A side information derived from the computation of $\theta(y^0)$ is the constraint of $(P_{\mathcal{F}})$ that y^0 most violates,

$$\mathcal{H}_+^0 := \{y \in \mathbb{R}^m \ : \ \langle w^0, y\rangle \geq \langle w^0, f(x(w^0))\rangle\}. \tag{12}$$

The positive half-space $\mathcal{H}_+^0$ supports $\mathcal{F}$ at $y = f(x(w^0))$ (because $f(x(w^0)) \in \mathcal{F}$ and $\mathcal{F}$ is contained in $\mathcal{H}_+^0$). The *deepest cut* produced by $\mathcal{H}_+^0$ in $\mathcal{F}^0$ generates the improved outer approximation of $\mathcal{F}$ described by $\mathcal{F}^1 = \mathcal{H}_+^0 \cap \mathcal{F}^0$. The minimizer y^1 of u subject to $y \in \mathcal{F}^1$ will be generally such that $\theta(y^1) > 0$ and an additional constraint $\mathcal{H}^1$ on y, the most violated by y^1, is incorporated into the relaxed problem. The repeated application of the previous steps leads to a global optimization algorithm for solving $(P_{\mathcal{F}})$.

Algorithm A_1

Step 0: Find $\mathcal{F}^0$ and set $k := 0$;
Step 1: Solve the generalized multiplicative problem

$$(P_{\mathcal{F}^k}) \left| \begin{array}{l} \text{minimize } u(y) = y_1 + \displaystyle\sum_{i=1}^{r} y_{2i}\, y_{2i+1} \\ \text{subject to} \quad y \in \mathcal{F}^k, \end{array} \right.$$

obtaining y^k;
Step 2: Find $\theta(y^k)$ by solving the maxmin subproblem (7) and (8). If $\theta(y^k) < \epsilon$, where $\epsilon > 0$ is a small tolerance, stop: y^k and $x(w^k)$ are ϵ-optimal solutions of $(P_{\mathcal{F}})$ and (P_{M}), respectively. Otherwise, define

$$\mathcal{F}^{k+1} := \{y \in \mathcal{F}^k \ : \ \langle w^k, y\rangle \geq \langle w^k, f(x(w^k))\rangle\},$$

set $k := k + 1$ and return to Step 1.

The proof of infinite convergence ($\epsilon = 0$) of algorithm A_1 to a global minimizer of $(P_{\mathcal{F}})$ is essentially the same provided in [21] for convex multiplicative problems: any subsequence $\{y^{k_l}\}$ of $\{y^k\}$ is such that $\theta(y^{k_l})$ tends to 0 as l tends to ∞. Additionally, it is readily seen that given $\epsilon > 0$ the algorithm terminates after finitely many iterations.

The quality of the ϵ-optimal solution y^ϵ at convergence of algorithm A_1 can be evaluated as follows. If $u^\star$ denotes the optimal value of $(P_{\mathcal{F}})$ (and (P_{M})), then

$$y_1^\epsilon + \sum_{i=1}^{r} y_{2i}^\epsilon y_{2i+1}^\epsilon \leq u^\star \leq f(x^\epsilon) + \sum_{i=1}^{r} f_{2i}(x^\epsilon) f_{2i+1}(x^\epsilon).$$

The first inequality derives from the fact that y^ϵ is a global minimizer of an outer approximation of $(P_{\mathcal{F}})$. The second one is due to the feasibility of the corresponding x^ϵ for (P_{M}). By using Theorem 2, the upper bound on $u^\star$ can be expressed as a function of ϵ:

$$\begin{aligned} u^\star &\leq \left(y_1^\epsilon + \theta(y^\epsilon)\right) + \sum_{i=1}^{r} \left(y_{2i}^\epsilon + \theta(y^\epsilon)\right)\left(y_{2i+1}^\epsilon + \theta(y^\epsilon)\right) \\ &\leq \left(y_1^\epsilon + \epsilon\right) + \sum_{i=1}^{r} \left(y_{2i}^\epsilon + \epsilon\right)\left(y_{2i+1}^\epsilon + \epsilon\right). \end{aligned}$$

The lower and upper bounds converge to $u^\star$ as ϵ tends to 0. The upper bound is more accurate when $w^\epsilon \in \mathbb{R}^m_{++}$, where w^ϵ is the maximizer of ϕ_{y^ϵ} over $\mathcal{W}$ and, by Theorem 2, the dual variables vector of (9). In this case one must have $f_i(x^\epsilon) = y_i^\epsilon + \theta(x^\epsilon)$ for $i = 1, 2, \ldots, m$, because all inequality constraints in (9) will be active. Theorem 2 and its geometric interpretation at the end of Sect. 3 also provide a justification for the convergence criterion of algorithm A_1. The scalar $\epsilon > 0$ establishes the maximum acceptable distance (measured by the infinity norm) between an approximate optimal solution and a feasible point of $(P_{\mathcal{F}})$.

4.1 Solving $(P_{\mathcal{F}^k})$ by constraint enumeration

Problem $(P_{\mathcal{F}^k})$ at Step 1 of algorithm A_1 can be rewritten as a linearly constrained quadratic programming problem of the form

$$(P_{\mathcal{F}^k}) \left| \begin{array}{l} \text{minimize } u(y) = \dfrac{1}{2} y^T Q y + c^T y \\ \text{subject to} \quad A^{(k)} y \geq b^{(k)}, \\ \qquad\qquad\quad \underline{y} \leq y \leq \overline{y}, \end{array} \right.$$

where

$$Q := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad c^T := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and $A^{(k)} \in \mathbb{R}^{k \times m}$, $b^{(k)} \in \mathbb{R}^k$, $\underline{y} \in \mathbb{R}^m$ and $\overline{y} \in \mathbb{R}^m$ characterize the matrix representation of $\mathcal{F}^k$. It is readily seen that $(P_{\mathcal{F}^k})$ is an indefinite quadratic program: the characteristic polynomial of Q is

$$\det(\lambda I - Q) = \lambda \underbrace{(\lambda^2 - 1) \cdots (\lambda^2 - 1)}_{r \text{ times}},$$

indicating that matrix Q has exactly r negative (equal to -1) and r positive (equal to 1) eigenvalues, where r is the number of products in the generalized multiplicative function. See [22] and [8] for comprehensive analyses of indefinite quadratic programming applications and methods. The characteristics of $(P_{\mathcal{F}^k})$ favour the application of the *constraint enumeration method*, discussed in details in [9] and [10].

It is known that an optimal solution of $(P_{\mathcal{F}^k})$ occurs at the boundary of $\mathcal{F}^k$ and can be found by constraint enumeration [10]. The number of constraints of $(P_{\mathcal{F}^k})$ is $2m + k$, where m is the number of functions in the generalized multiplicative objective and k is the current number of (deepest) cuts generated by algorithm A_1.

In principle there are 2^{2m+k} ways of enumerating the constraints of $(P_{\mathcal{F}^k})$, each one leading to a Karush-Kuhn-Tucker system solved in at most $O((3m + k)^3)$ arithmetic operations [10]. However, since the lower and upper bound constraints of any component of y can not be active at the same time, the number of possible combinations is smaller. In addition, only subsets of at least r constraints need to be checked.

Theorem 3 *At least r of the constraints are active at any (local) global solution point of* $(P_{\mathcal{F}^k})$.

Proof See [9]. □

When a new constraint is incorporated into $\mathcal{F}^k$, only combinations of at least r constraints *that include the new one* need to be investigated. The reason is that the optimal solution of $P_{\mathcal{F}^k}$ must have been declared infeasible, that is, it does not solve the generalized multiplicative problem. Supposing that algorithm A_1 converges after N iterations, it is possible to show that the latter property avoids considering 2^N combinations. Computational experiences reported in Sect. 5 indicate that N is usually small, which enables an efficient resolution of $P_{\mathcal{F}^k}$ by constraint enumeration.

5 Computational experiences

Consider the illustrative example discussed in [16], where an alternative algorithm for generalized convex multiplicative programming is developed and tested. (The method derived in [16] transforms (P_M) into a concave minimization problem, which is consistent with the fact that indefinite quadractic problems can be formulated and solved by concave minimization. See [22].) The data involved are: $n = 2$, $p = 5$, $r = 2$ and

$$\begin{aligned}
&f_1(x) = 3x_1 - 4x_2 + 15,\;\; f_2(x) = x_1 + 2x_2 - 1.5,\;\; f_3(x) = 2x_1 - x_2 + 4,\\
&f_4(x) = x_1 - 2x_2 + 8.5,\;\; f_5(x) = 2x_1 + x_2 - 1,\\
&g_1(x) = 5x_1 - 8x_2 \geq -24,\;\; g_2(x) = 5x_1 + 8x_2 \leq 44,\\
&g_3(x) = 6x_1 - 3x_2 \leq 15,\;\; g_4(x) = 4x_1 + 5x_2 \geq 10,\;\; g_5(x) = x_1.
\end{aligned}$$

It can be shown that f_1, f_2, f_3, f_4 and f_5 are positive over Ω. Letting, as in [16], $\underline{y} = (3, 1, 1, 2, 2)$ and $\overline{y} = (22.5, 9, 9, 11, 11)$, one obtains the results reported in Table 1.

Algorithm A_1 converged after only two iterations to the global optimal solution $x^\star = (0, 3)$, the same found in [16] (in an unspecified number of iterations).

The main objective of this section is to investigate the computational performance of the proposed global optimization algorithm. More extensive tests were conducted on the basis of the following class of generalized multiplicative problems:

$$(P_M)\quad \left|\begin{array}{l} \text{minimize } \langle c^1, x\rangle + \dfrac{1}{2}x^T C^T C x + \displaystyle\sum_{i=1}^{r} \langle c^{2i}, x\rangle\langle c^{2i+1}, x\rangle \\ \text{subject to } \quad Ax \geq b, \quad x \in \mathbb{R}^n_+, \end{array}\right.$$

where $A \in \mathbb{R}^{p\times n}$, $b \in \mathbb{R}^p$, $C \in \mathbb{R}^{n\times n}$ and $c^i \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$, are constant matrices with pseudo-randomly entries generated in the interval [0, 100].

Algorithm A_1 was coded in MATLAB (V. 6.1)/Optimization Toolbox (V. 2.1.1) [20] and runs on a Pentium IV, 2.4GHz, 512MB RAM personnal computer. The tolerance for convergence was fixed at $\epsilon = 10^{-5}$.

The following indices characterize the performance of algorithm A_1: W, number of convex minimizations needed for solving the maxmin subproblem (7) and (8); C, number of cuts needed for convergence; T, CPU time (in seconds).

Table 1 Convergence of Algorithm A_1

k	y^k	w^k	$x(w^k)$	$\theta(y^k)$
0	(3,1,1,2,2)	(0.3333,0.6667,0,0,0)	(0,2)	2.3333
1	(3,3,1,9,2)	(1,0,0,0,0)	(0,3)	0.0000

Table 2 Average values of W, C and T for $r = 1$

n	20	20	50	50	100	100	150
p	10	30	30	70	70	130	130
C	3.36 (0.67)	3.11 (0.78)	13.50 (5.27)	14.90 (3.48)	15.88 (0.33)	15.70 (0.48)	15.30 (0.48)
W	14.54 (4.03)	14.33 (4.92)	22.20 (4.23)	25.70 (1.16)	25.02 (2.12)	25.70 (2.01)	24.90 (0.98)
T	0.39 (0.14)	0.45 (0.15)	75.45 (38.77)	86.04 (29.05)	98.32 (14.29)	92.42 (21.50)	95.16 (22.27)

Table 3 Average values of W, C and T for $r = 2, 3$

r	2	2	2	3	3	3
n	20	50	50	20	50	50
p	30	30	70	30	30	70
C	11.30 (2.21)	12.10 (0.32)	11.90 (0.32)	8.40 (0.52)	8.25 (0.46)	8.33 (0.46)
W	34.20 (5.14)	41.70 (5.92)	44.90 (5.57)	39.60 (7.39)	58.37 (9.85)	70.88 (11.95)
T	57.78 (19.73)	67.42 (1.68)	70.33 (5.57)	74.14 (18.30)	77.64 (16.12)	82.64 (17.28)

Ten problems for selected combinations of n (number of variables) and p (number of constraints) were solved. Average and standard deviation values (in parenthesis) of C, W and T are presented.

Table 2 reports the results of algorithm A_1 for $r = 1$. On average problem P_W has to be solved W times in order to produce C deepest cuts, but the effort spent in this task is worthwhile: the number of constraints involved in the enumeration procedure of Step 1 is kept small.

Instances with $r = 1$ may be difficult to solve because the indefinite matrix Q must have exactly one negative eigenvalue when the global minimum occurs at an interior point of a face of $\mathcal{F}^k$ [22].

A critical parameter for evaluating the performance of generalized multiplicative programming algorithms is the number of products in the objective function. The results of algorithm A_1 for $r = 2$ and $r = 3$ are presented in Table 3.

Tables 2 and 3 show that n and p have little influence on C, the number of cuts generated by the algorithm in the outcome space, and a substantial influence on W (and consequently on T) because W is related to the resolution of optimization problems in $\mathbb{R}^n$. Tables 2 and 3 also show that becomes increasingly more difficult to identify deepest cuts: the ratio W/C increases as r increases. However the number of deepest cuts tends to decrease as the number of constraints—at least r—activated by global minimizers increases.

The class of generalized multiplicative problems and test conditions under which computational experiences were conducted and reported in this section are the same established in [16]. Since the results of Table 2 and those provided in [16] were obtained by using different computational resources, the following relative performance measure is adopted:

$$\tau_{i,j} := \frac{\text{average time for } n = i \text{ and } p = j}{\text{average time for } n = 20 \text{ and } p = 10}. \tag{13}$$

The growths of the computing times requirements of the algorithms are presented in Table 4.

A comparison between the two algorithms indicates that although initially the computing time requirements of algorithm A_1 grow faster due to higher computational costs for

Table 4 Growths of computing times requirements for $r = 1$

	$\tau_{20,30}$	$\tau_{50,30}$	$\tau_{50,70}$	$\tau_{100,70}$	$\tau_{100,130}$	$\tau_{150,130}$
Algorithm of [16]	3.800	17.40	55.80	167.0	576.4	964.4
Algorithm A_1	1.154	193.5	220.6	252.1	237.0	244.0

obtaining deepest cuts and solve $(P_{\mathcal{F}^k})$ by constraint enumeration, its growth rate tends to be substantially smaller than that exhibited by the algorithm of [16] as the number of variables and constraints increases.

By analysing Tables 2 and 4 one observes that the average number of deepest cuts, C, and therefore the computational effort needed for solving $(P_{\mathcal{F}^k})$, varies substantially when (n, p) goes from (20, 30) to (50, 30). On the other hand, C varies slowly from (50, 30) to (150, 130), which indicates that the growth in CPU time of algorithm A_1 must be associated to a growth in CPU time for generating deepest cuts. It happens that C deepest cuts are derived from W ordinary cuts and the difference between W and V, the average number of vertices needed to produce cuts according to the algorithm of [16], is small when (n, p) goes from (50, 30) to (150, 130). However, while algorithm A_1 generates each ordinary cut by solving (P_W), that is, by minimizing a convex combination of $f_1, f_2, \ldots, f_m$ $(m = 2r + 1)$ subject to the constraints of (P_{M}), each vertex generated by the algorithm of [16] requires the minimization of a non-negative combination of $f_1, f_2^2, \ldots, f_m^2$ subject to the same constraints. In the present computational experiments both programs are convex and quadratic (because f_1 is quadratic and $f_2, f_3, \ldots, f_m$ are linear functions), but the growth in the computing time requirements for solving the former is slower as n or p increases.

The behavior observed in Table 4 tends to be even more pronounced as r increases. When $r = 3$ and $(n, p) = (50, 70)$, for example, one obtains $W = 70.88$ and $V = 3{,}896.6$. Under such circumstances, the computational effort demanded by the algorithm of [16] is much higher than that exhibited by algorithm A_1.

6 Conclusions

A global optimization approach for generalized convex multiplicative programs was proposed in this paper. By using convex analysis results the original problem was reformulated in the outcome space as a linearly constrained semi-infinite equivalent problem, and then solved through relaxation. Each relaxation—an outer approximation of the problem—was expressed as a linearly constrained indefinite quadratic program with special characteristics. By working exclusively with the linear inequalities (generated by maxmin subproblems) that produce deepest cuts in the outcome space, the number of constraints needed for the convergence of the algorithm was kept small and the sequence of indefinite quadratic programs could be solved efficiently by constraint enumeration.

Computational experiences have attested the viability and efficiency of the proposed global optimization algorithm, which is, in addition, easily programmed through standard optimization packages.

Further properties of the proposed approach as well as its extension to other classes of multiplicative and fractional global optimization problems are under current investigation.

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